Problem Set 3 – Relativity (`GOI36A`)

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1 Actions, Part 2: Electric Boogaloo

Remember from last week that, given a Lagrangian $\mathcal{L}$ for a scalar field Φ, the equation of motion governing this field (more formally known as the Euler-Lagrange equation) is given by

$$\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \right) = 0 \ . \tag{1.1}$$

This formula is very general, and it applies to any system where we can write down a Lagrangian $\mathcal{L}(\Phi, \partial_\mu \Phi, x^\mu)$ that is a function of the field Φ as well as its derivatives $\partial_\mu \Phi$ at any given point x^μ on the manifold of consideration.

This week, we will look at an action that describes a photon in a four-dimensional Minkowski spacetime. The dynamics of this photon are encoded into a vector field A_μ that lives on the Minkowski spacetime. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + 4\pi A_\mu J^\mu \ , \tag{1.2}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is commonly referred to as the *field strength tensor*, and J^μ is a background field that interacts with A_μ. (For example, if we wanted to ask what happens to a photon when we apply a constant background magnetic field, we would use J^μ to specify this background magnetic field.) In this problem, we will explore the dynamics of the photon field A_μ using the Lagrangian formulation.

1.) Using the definition of the field strength, show that the field strength tensor satisfies

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0 \ . \tag{1.3}$$

This is called the *Bianchi identity.*

2.) Show that the Bianchi identity is trivial unless the indices μ, ν, and ρ all take different values. Based on this, how many non-trivial equations does the Bianchi identity represent?

3.) Since we are looking at a vector field and not a scalar field, we cannot just use the scalar field Euler-Lagrange equation (1.1). Instead, we need a generalization that will work for a vector field. Argue that the most natural generalization is as follows:

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) = 0 \ . \tag{1.4}$$

Is it important how we have chosen the indices in this expression? Why or why not?

4.) When we say that the field J^μ is a background field, this means that we have in mind that we are fixing what values J^μ takes at every point in our spacetime. Since we have fixed it, we do not vary our Lagrangian with respect to J^μ; A_μ is the only field that we want to vary our Lagrangian with respect to the Euler-Lagrange equations (1.4) for a vector field.

With this in mind, show that the Euler-Lagrange equations for the field A_μ are given by

$$\partial_\mu F^{\mu\nu} = -4\pi J^\nu \ . \tag{1.5}$$

5.) Even if you don't know it yet, what you've actually just done is derive Maxwell's equations! In units where $\mu_0 = \varepsilon_0 = c = 1$, Maxwell's equations can be written as

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= 4\pi\rho \ , \\ \vec{\nabla} \times \vec{B} - \partial_t \vec{E} &= 4\pi \vec{J} \ , \\ \vec{\nabla} \cdot \vec{B} &= 0 \ , \\ \vec{\nabla} \times \vec{E} + \partial_t \vec{B} &= 0 \ . \end{aligned} \tag{1.6}$$

where $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields (respectively), ρ is the electric charge density, and $\vec{J}$ is the electric current density. Show that, if the field strength $F^{\mu\nu}$ and background source J^μ are given in component form by

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix} \ , \quad J^\mu = (\rho \, , J_1 \, , J_2 \, , J_3) \ , \tag{1.7}$$

then the equations you derived in problems 1 and 3 are exactly equivalent to Maxwell's equations. Specifically, you should find that the equations of motion from problem 4 are equivalent to the first two lines in (1.6), and the Bianchi identity from problem 1 is equivalent to the last two lines of (1.6). Then, argue that (1.7) is the most general form possible for the tensors $F^{\mu\nu}$ and J^μ. (For this last part, you'll need to think about symmetries and how many independent components these tensors have.)

6.) We can define the *electromagnetic stress tensor* $T^{\mu\nu}$ in terms of the field strength as follows:

$$T^{\mu\nu} \equiv F^{\mu}{}_{\rho} F^{\nu\rho} - \frac{1}{4}\eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \ . \tag{1.8}$$

Use your results from the previous problems to prove that

$$\partial_\mu T^{\mu\nu} = 4\pi F^{\mu\nu} J_\mu \ . \tag{1.9}$$

2 Spacelike, Timelike, and Null Vectors

For any vector V in a Minkowski spacetime, we can classify it into one of three categories based on its norm $||V||^2 \equiv V_\mu V^\mu$:

$$\begin{aligned} V_\mu V^\mu < 0 &\implies V^\mu \text{ is timelike} \ , \\ V_\mu V^\mu = 0 &\implies V^\mu \text{ is null} \ , \\ V_\mu V^\mu > 0 &\implies V^\mu \text{ is spacelike} \ . \end{aligned} \tag{2.1}$$

In the following questions, let T be a timelike vector, S a spacelike vector, and U and V two null vectors in a d-dimensional spacetime. In what follows, you may find the Schwarz inequality useful:

$$(\vec{x} \cdot \vec{y})^2 \leq (\vec{x} \cdot \vec{x})(\vec{y} \cdot \vec{y}) \ . \tag{2.2}$$

7.) Let X be some vector such that $T_\mu X^\mu = 0$. Prove that X must be spacelike.

8.) Suppose there is an inertial frame in which $S^\mu = (0, \vec{s})$ and $T^\mu = (t^0, \vec{0})$. Then, prove that S and T are orthogonal in all inertial frames, i.e. that $S_\mu T^\mu = 0$ in all intertial frames.

9.) **Bonus:** Prove the converse of the above statement: if T and S are orthogonal ($S_\mu T^\mu = 0$), then there exists an inertial frame in which $S^\mu = (0, \vec{s})$ and $T^\mu = (t^0, \vec{0})$.

10.) Prove that if U and V are orthogonal, then they are proportional to one another.

3 The Minkowski Metric in Other Coordinates

As should be familiar by now, the four-dimensional Minkowski metric can be written as

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \; . \tag{3.1}$$

In this problem, we will investigate what happens to this metric when we make coordinate transformations.

11.) First, one of the simplest changes of coordinates is to go to polar coordinates (t, r, ϕ, z) by making the following changes of variables:

$$x = r \cos\phi \; , \quad y = r \sin\phi \; . \tag{3.2}$$

Derive the form of the metric in polar coordinates.

12.) Another commonly used set of coordinates are spherical coordinates (t, r, θ, ϕ), which are functions of the Cartesian coordinates as follows:

$$x = r \sin\theta \, \sin\phi \; , \quad y = r \sin\theta \, \cos\phi \; , \quad z = r \cos\theta \; . \tag{3.3}$$

Derive the form of the metric in spherical coordinates.